High-energy QCD and Wilson lines

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JLAB & ODU

LANL Nuclear Theory Seminar 13 March 2014

Outline

1. Introduction: BFKL pomeron in hign-energy pQCD
   - Regge limit in QCD.
   - Perturbative QCD at high energies.
   - BFKL and collider physics
2. High-energy scattering and Wilson lines
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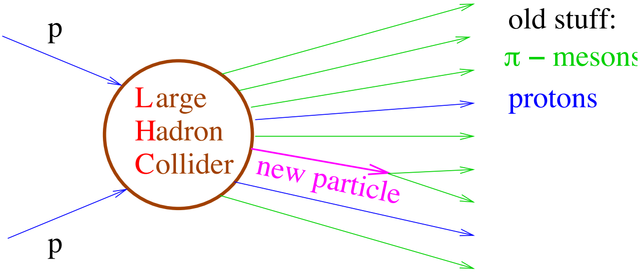
   - Evolution equation for color dipoles.
   - Light-ray vs Wilson-line operator expansion.
   - Leading order: BK equation.
3. NLO high-energy amplitudes
   - Conformal composite dipoles and NLO BK kernel in $\mathcal{N} = 4$.
   - NLO amplitude in $\mathcal{N} = 4$ SYM
   - Photon impact factor.
   - NLO BK kernel in QCD.
   - rcBK.
   - NLO hierarchy of Wilson-lines evolution.
   - Conclusions

Light-ray operators

Heisenberg uncertainty principle: $\Delta x = \frac{\hbar}{p} = \frac{\hbar c}{E}$

LHC: E=7 $\rightarrow$ 14 TeV $\Leftrightarrow$ distances $\sim 10^{-18}$ cm

(Planck scale is 10^{-33} cm - a long way to go!)



To separate a “new physics signal” from the “old” background one needs to understand the behavior of QCD cross sections at large energies

Strong interactions at asymptotic energies: Froissart bound

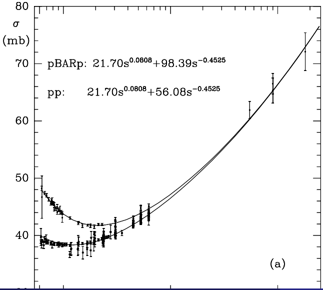
Regge limit: $E \gg$ everything else

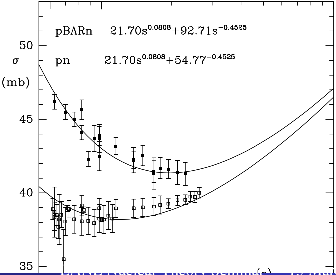
$$\left.\begin{array}{c}\text{Causality}\\ \text{Unitarity}\end{array}\right\} \quad\Rightarrow\quad \sigma_{\rm tot} \overset{E\to\infty}{\leq} \ln^2 E$$

Froissart, 1962

Long-standing problem - not explained in any quantum field theory (or string theory) in 50 years!

Experiment: $\sigma_{\rm tot} \sim s^{0.08}$ ($s \equiv 4E^2_{\rm c.m.}$). Numerically close to $\ln^2 E$.





Deep inelastic scattering in QCD

$D_q(x_B) \to D_q(x_B, Q^2)$ - "scaling violations"

DGLAP evolution (LLA(Q^2))

$$Q\frac{d}{dQ}D_q(x, Q^2) = \int_x^1 dx' K_{\rm DGLAP}(x, x') D_q(x', Q^2)$$

Dokshitzer, Gribov, Lipatov, Altarelli, Parisi, 1972-77

$$K_{\rm DGLAP} = \alpha_s(Q)K_{\rm LO} + \alpha_s^2(Q)K_{\rm NLO} + \alpha_s^3(Q)K_{\rm NNLO}...$$

The DGLAP equation sums up logs of $\frac{Q^2}{m_N^2}$

$$D_q(x, Q^2) = \sum_n \left(\alpha_s \ln\frac{Q^2}{m_N^2}\right)^n \left[a_n(x) + \alpha_s b_n(x) + \alpha_s^2 c_n(x) + ...\right]$$

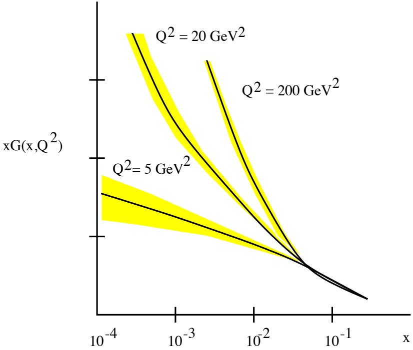
One fit at low $Q_0^2 \sim 1$ GeV^2 describes all the experimental data on DIS!

Regge limit in DIS: $E \gg Q \equiv\ x_B \ll 1$

DGLAP evolution $\equiv\ Q^2$ evolution

$$Q\frac{d}{dQ}D_g(x_B,Q^2) = K_{\mathrm{DGLAP}}D_g(x_B,Q^2)$$

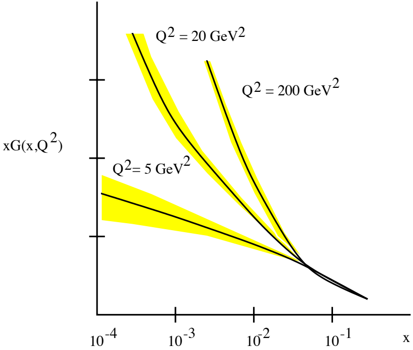
HERA data for $xD_g(x)$



Not really a theory -
needs the x-dependence of the input
at $Q_0^2\ \sim\ 1\mathrm{GeV}^2$

Deep inelastic scattering at small x_B

HERA data for $xD_g(x)$



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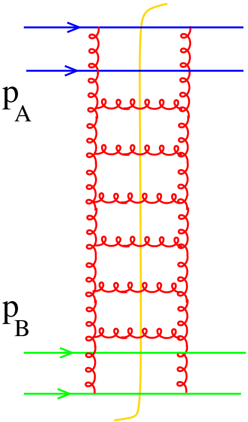
BFKL evolution $\equiv \; x_B$ evolution
(Balitsky, Fadin, Kuraev, Lipatov, 1975-78)

$$\frac{d}{dx_B}D_g(x_B, Q^2) = K_{\rm BFKL}D_g(x_B, Q^2)$$

Theory, but with problems

In pQCD: Leading Log Approximation ⇒ BFKL pomeron

$$s = (p_A + p_B)^2 \simeq 4E^2$$



Leading Log Approximation (LLA(x)):

$$\alpha_s \ll 1, \quad \alpha_s \ln s \sim 1$$

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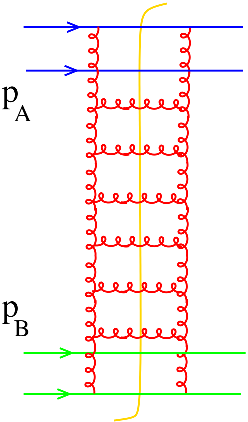

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The sum of gluon ladder diagrams gives

$$\sigma_{\rm tot} \sim s^{12\frac{\alpha_s}{\pi}\ln 2}$$ BFKL pomeron

Numerically: for DIS at HERA

$$\sigma \sim s^{0.3} = x_B^{-0.3}$$

- qualitatively OK

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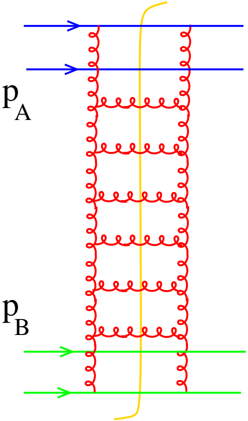

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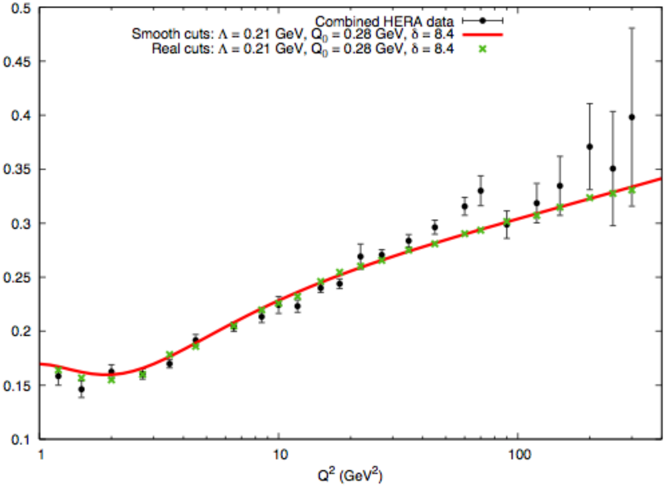
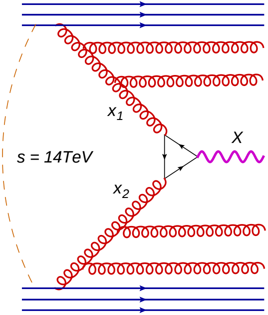
BFKL *vs* HERA data

$$F_2(x_B, Q^2) \;=\; c(Q^2)\, x_B^{-\lambda(Q^2)}$$



M.Hentschinski, A. Sabio Vera and C. Salas, 2010

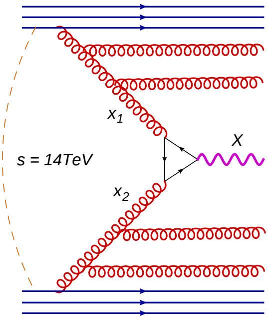
DGLAP vs BFKL in particle production



Collinear factorization (LLA(Q^2)):

$$\sigma_{pp\to X} = \int_0^1 dx_1 dx_2 D_g(x_1, m_X) D_g(x_2, m_X) \sigma_{gg\to X}$$

sum of the logs $\left(\alpha_s \ln \frac{m_X^2}{m_N^2}\right)^n, \quad \ln \frac{s}{m_X^2} \sim 1$



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LLA(x): k_T-factorization

$$\sigma_{pp\to X} = \int dk_1^\perp dk_2^\perp g(k_1^\perp, x_A) g(k_2^\perp, x_B) \sigma_{gg\to X}$$

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Much less understood theoretically.

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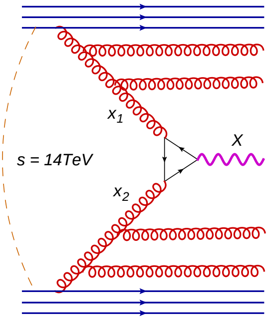

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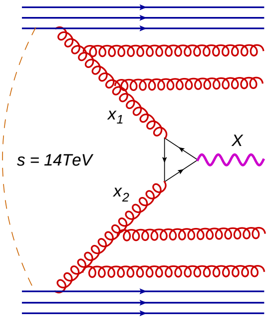
Much less understood theoretically.

For Higgs production in the central rapidity region $x_{1.2} \sim \frac{m_H}{\sqrt{s}} \simeq 0.01$ and we know from DIS experiments that at such x_B the DGLAP formalism works pretty well $\Rightarrow$ no need for BFKL resummation

DGLAP vs BFKL in particle production



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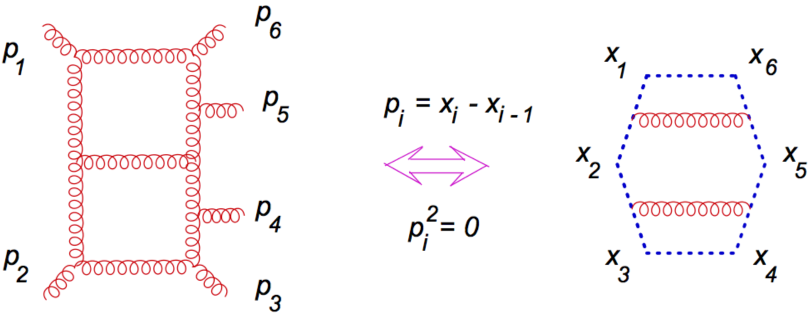
Much less understood theoretically.

For $m_X \sim 10$GeV (like $\bar{b}b$ pair or mini-jet) collinear factorization does not seem to work well $\Rightarrow$ some kind of BFKL resummation is needed.

Uses of BFKL: MHV amplitudes in $\mathcal{N} = 4$ SYM

MHV gluon amplitudes $\Leftrightarrow$ light-like Wilson-loop polygons

Alday, Maldacena (at large $\alpha_s N_c$)



Checked up to 6 gluons/2 loops (Korchemsky et. al).

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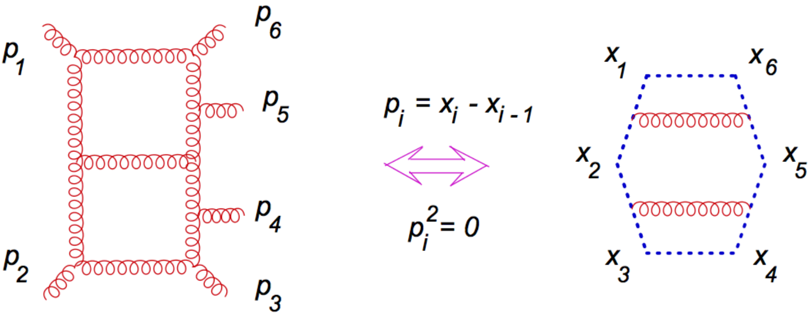

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BDS ansatz: $\ln A^{\text{MHV}} =$ IR terms $+ F_n, \quad F_n = \Gamma_{\text{cusp}}(\text{angles}) + (F_n^{1)} + R_n)$

BFKL in multi-Regge region ⇒ asymptotics of remainder function R_n (Lipatov et a)l

Uses of BFKL: Anomalous dimensions of twist-2 operators

Structure functions of DIS are determined by matrix elements of twist-2 operators

$$\mathcal{O}_G^{(j)} = F_{\mu_1 \xi} D_{\mu_2} \ldots D_{\mu_{j-2}} F_{\mu_j}^{\ \xi}$$

$$\mu^2 \frac{d}{d\mu^2} \mathcal{O}_G^{(j)} = \frac{\gamma_{(j)}(\alpha_s)}{4\pi} \mathcal{O}_G^{(j)}$$

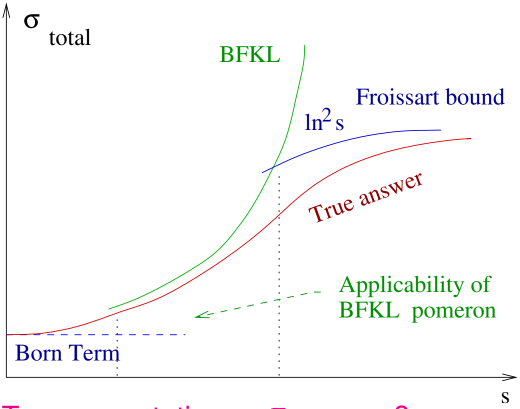
BFKL gives asymptotics of $\gamma_{(j)}$ at $j \to 1$ in all orders in α_s

$$\gamma_{(j)} = \sum_n \left(\frac{\alpha_s}{j-1}\right)^n \left[C^{(n)}_{\rm LO\ BFKL} + \alpha_s C^{(n)}_{\rm NLO\ BFKL}\right]$$

Checked by explicit calculation of Feynman diagrams.up to 3 loops in QCD and $\mathcal{N} = 4$ SYM. (Janik et al)

Integrablility of spin chains corresponding to evolution of $\mathcal{N} = 4$ SYM operators $\Rightarrow \gamma_{(j)}$ in 5 loops agrees with BFKL (Janik et al).
For all order of pert. theory: Y-system of equations (Gromov, Kazakov, Viera). Hopefully agrees with BFKL.

Towards the high-energy QCD

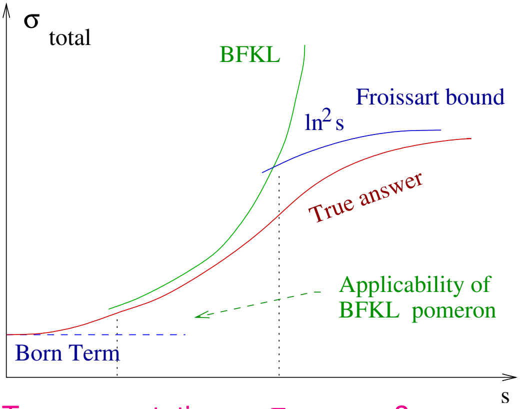


$\sigma_{\text{tot}} \sim s^{12\frac{\alpha_s}{\pi}\ln 2}$ violates Froissart bound $\sigma_{\text{tot}} \leq \ln^2 s$ $\Rightarrow$ pre-asymptotic behavior.

True asymptotics as $E \to \infty$ = ?

Possible approaches:

- Sum all logs $\alpha_s^m \ln^n s$
- Reduce high-energy QCD to $2+1$ effective theory



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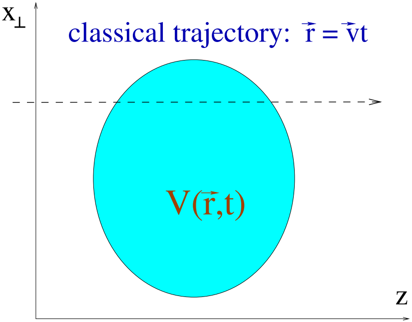
This talk: NLO corrections $\alpha_s^{n+1} \ln^n s$

High-energy scattering and "Wilson lines" in quantum mechanics



WKB approximation: $\Psi \sim e^{\frac{i}{\hbar}S}$

$$S = \int (pdz - Edt)$$

$$= -Et + \int^{z} dz' \sqrt{2m(E - V(z'))}$$

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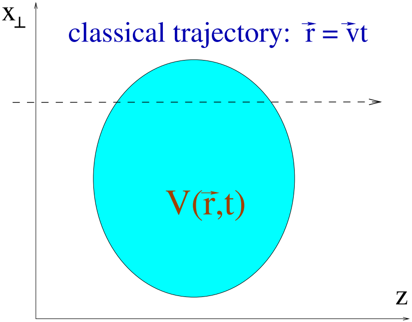

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High energy: $E \gg V(x) \Rightarrow$

$$\Psi(\vec{r}, t) = e^{-\frac{i}{\hbar}(Et - kx)} \; e^{-\frac{i}{v\hbar}\int_{-\infty}^{z} dz' V(z')}$$

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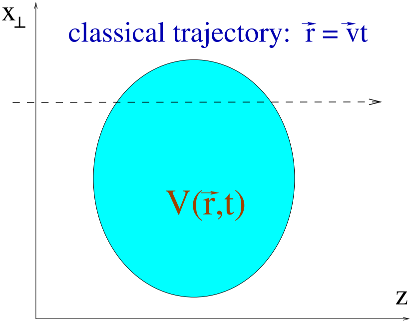

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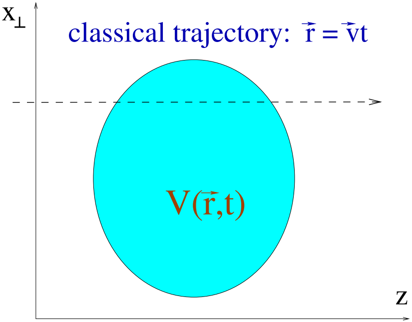

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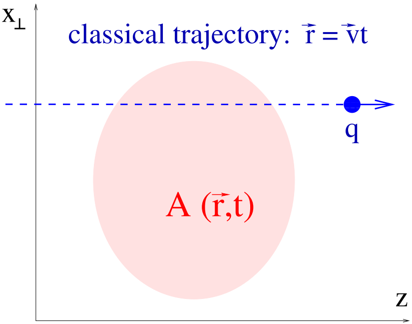
The scattering amplitude is proportional to $\Psi(t = \infty)$ defined by

$$U(x_\perp) \ = \ e^{-\frac{i}{v\hbar}\int_{-\infty}^{\infty} dz' V(z' + x_\perp)}$$

Glauber formula: $\sigma_{\rm tot} = 2\int d^2x_\perp \ [1 - \Re U(x_\perp)]$

High-energy phase factor in QED and QCD



$$
\begin{aligned}
S_{\rm e} &= \int dt\,\Big\{-mc^2\sqrt{1-\frac{v^2}{c^2}} - e\Phi + \frac{e}{c}\vec{v}\cdot\vec{A}\Big\} \\
&= S_{\rm free} + \int dt\Big(-e\Phi + \frac{e}{c}\vec{v}\cdot\vec{A}\Big)
\end{aligned}
$$

$\Rightarrow$ phase factor for the high-energy scattering is

$$
\begin{aligned}
U(x_\perp) &= e^{-\frac{ie}{\hbar c}\int_{-\infty}^{\infty}dt(-e\Phi+\frac{e}{c}\vec{v}\cdot\vec{A})} \\
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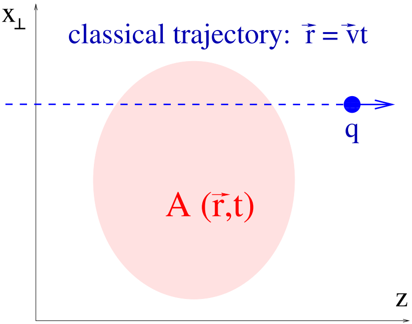
High-energy phase factor in QED and QCD



$$S_e = \int dt \left\{ -mc^2\sqrt{1-\frac{v^2}{c^2}} - e\Phi + \frac{e}{c}\vec{v}\cdot\vec{A} \right\}$$

$$= S_{\text{free}} + \int dt\left(-e\Phi + \frac{e}{c}\vec{v}\cdot\vec{A}\right)$$

$\Rightarrow$ phase factor for the high-energy scattering is

$$U(x_\perp) = e^{-\frac{ie}{\hbar c}\int_{-\infty}^{\infty} dt(-e\Phi + \frac{e}{c}\vec{v}\cdot\vec{A})}$$

$$= e^{-\frac{ie}{\hbar c}\int_{-\infty}^{\infty} dt\ \dot{x}_\mu A^\mu(x(t))}$$

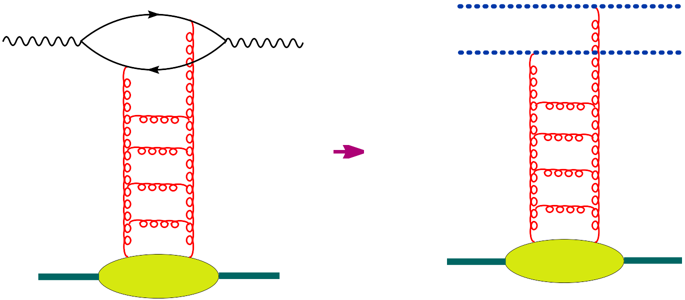
In QCD $e \to -g$, $A_\mu \to A_\mu \equiv A^a_\mu t^a$ $\qquad$ t^a - color matrices

$$\Rightarrow\ U(x_\perp, v) = P\exp\{\frac{ig}{\hbar c}\int_{-\infty}^{\infty} dt\ \dot{x}_\mu A^\mu(x(t))\} \qquad \text{Wilson} - \text{line operator}$$

(Later $\hbar = c = 1$)

DIS at high energy

- At high energies, particles move along straight lines ⇒ the amplitude of $\gamma^* A \to \gamma^* A$ scattering reduces to the matrix element of a two-Wilson-line operator (color dipole):



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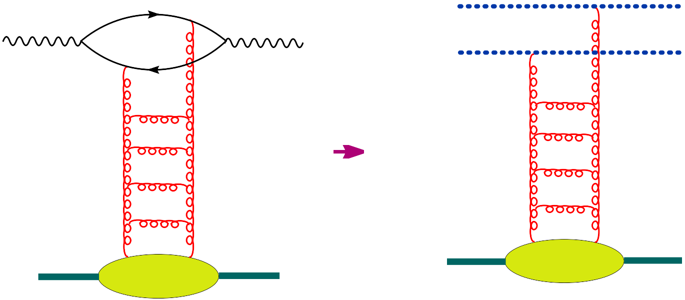
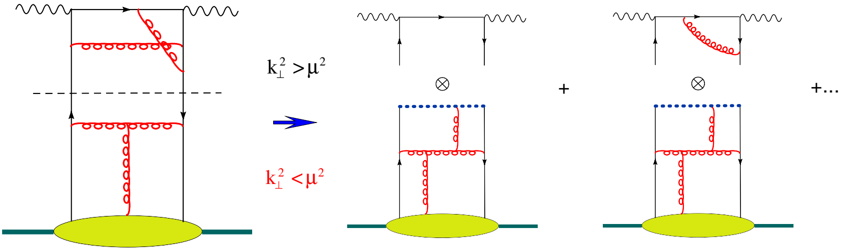


$$A(s) = \int \frac{d^2k_\perp}{4\pi^2} I^A(k_\perp) \langle B|\mathrm{Tr}\{U(k_\perp)U^\dagger(-k_\perp)\}|B\rangle$$

Formally, ➔ means the operator expansion in Wilson lines

Light-cone expansion and DGLAP evolution in the NLO



μ^2 - factorization scale (normalization point)

$k_\perp^2 \ > \ \mu^2$ - coefficient functions
$k_\perp^2 \ < \ \mu^2$ - matrix elements of light-ray operators (normalized at μ^2)

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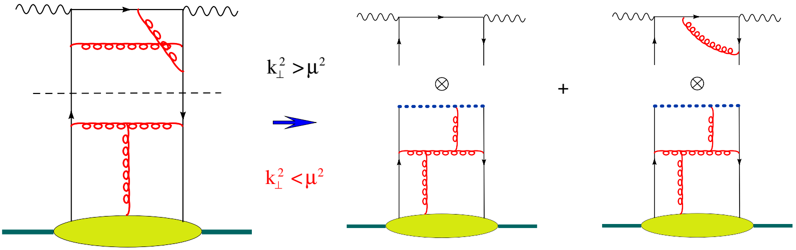

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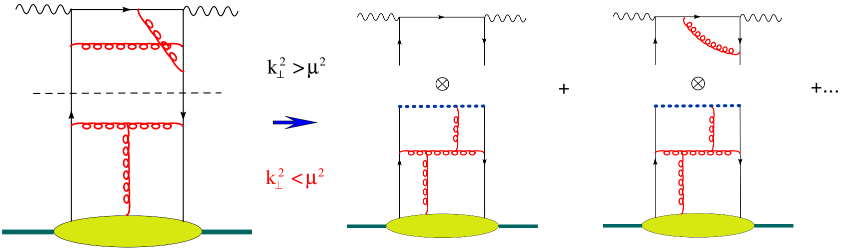
OPE in light-ray operators $(x-y)^2 \to 0$

$$T\{j_\mu(x)j_\nu(0)\} \;=\; \frac{x_\xi}{2\pi^2x^4}\Big[1+\frac{\alpha_s}{\pi}(\ln x^2\mu^2+C)\Big]\bar\psi(x)\gamma_\mu\gamma^\xi\gamma_\nu[x,0]\psi(0)+O(\frac{1}{x^2})$$

$$[x,y] \;\equiv\; Pe^{ig\int_0^1 du\;(x-y)^\mu A_\mu(ux+(1-u)y)}$$ - gauge link

Light-cone expansion and DGLAP evolution in the NLO



μ^2 - factorization scale (normalization point)

$k_\perp^2 \ > \ \mu^2$ - coefficient functions
$k_\perp^2 \ < \ \mu^2$ - matrix elements of light-ray operators (normalized at μ^2)

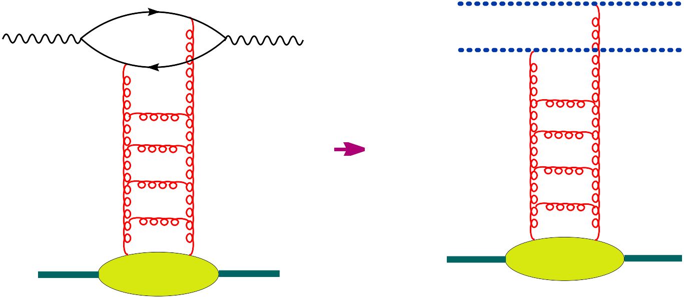
Renorm-group equation for light-ray operators ⇒ DGLAP evolution of parton densities $(x-y)^2 = 0$

$$\mu^2 \frac{d}{d\mu^2}\bar\psi(x)[x,y]\psi(y) \ = \ K_{\rm LO}\bar\psi(x)[x,y]\psi(y) + \alpha_s K_{\rm NLO}\bar\psi(x)[x,y]\psi(y)$$

- Factorize an amplitude into a product of coefficient functions and matrix elements of relevant operators.
- Find the evolution equations of the operators with respect to factorization scale.
- Solve these evolution equations.
- Convolute the solution with the initial conditions for the evolution and get the amplitude

DIS at high energy: relevant operators

- At high energies, particles move along straight lines $\Rightarrow$
  the amplitude of $\gamma^* A \to \gamma^* A$ scattering reduces to the matrix element of a two-Wilson-line operator (color dipole):



$$A(s) = \int \frac{d^2 k_\perp}{4\pi^2} I^A(k_\perp) \langle B|\mathrm{Tr}\{U(k_\perp)U^\dagger(-k_\perp)\}|B\rangle$$

$$U(x_\perp) = \mathrm{Pexp}\Big[ig\int_{-\infty}^{\infty} du\ n^\mu A_\mu(un + x_\perp)\Big] \qquad \text{Wilson line}$$

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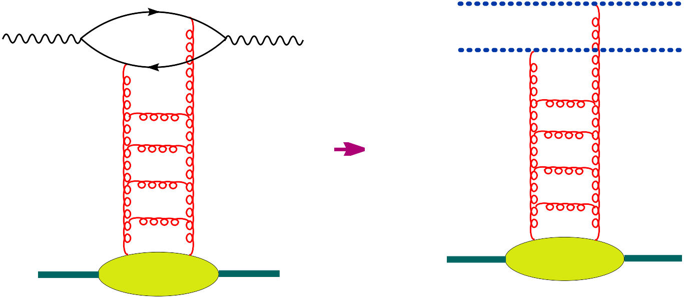

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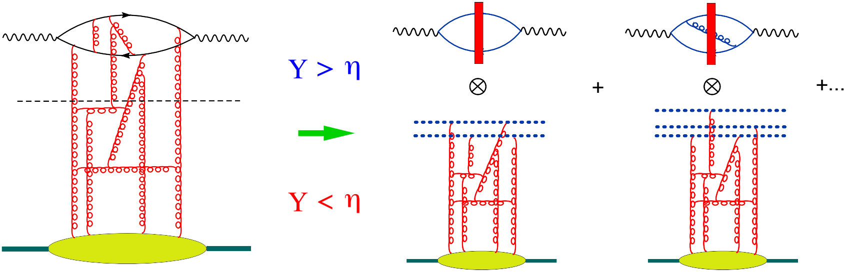
Formally, ➔ means the operator expansion in Wilson lines



η - rapidity factorization scale

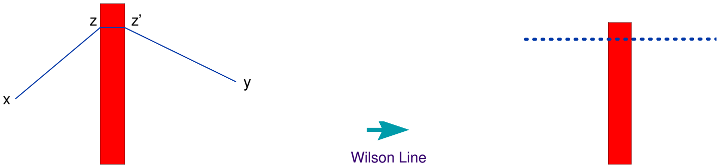
Rapidity Y > η - coefficient function ("impact factor")

Rapidity Y < η - matrix elements of (light-like) Wilson lines with rapidity divergence cut by η

$$U_x^\eta = \text{Pexp}\Big[ig\int_{-\infty}^{\infty} dx^+ A_+^\eta(x_+, x_\perp)\Big]$$

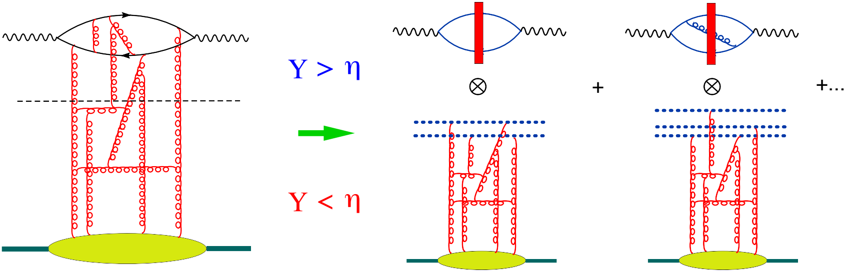
$$A_\mu^\eta(x) = \int \frac{d^4k}{(2\pi)^4}\theta(e^\eta - |\alpha_k|)e^{-ik\cdot x}A_\mu(k)$$

Each path is weighted with the gauge factor $Pe^{ig\int dx_\mu A^\mu}$. Quarks and gluons do not have time to deviate in the transverse space $\Rightarrow$ we can replace the gauge factor along the actual path with the one along the straight-line path.



[$x \to z$: free propagation]$\times$
$[U^{ab}(z_\perp)$ - instantaneous interaction with the $\eta < \eta_2$ shock wave$]\times$
[$z \to y$: free propagation]

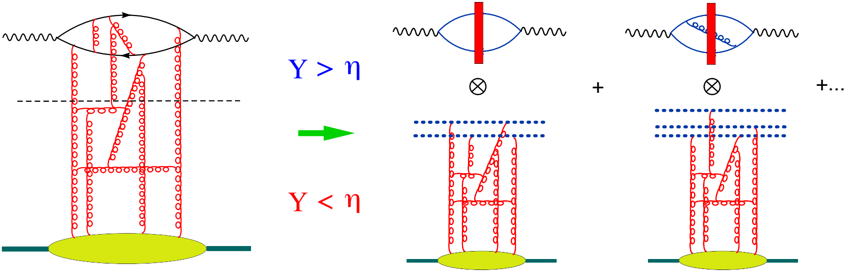
High-energy expansion in color dipoles



The high-energy operator expansion is

$$T\{\hat{j}_\mu(x)\hat{j}_\nu(y)\} = \int d^2z_1 d^2z_2 \; I^{\rm LO}_{\mu\nu}(z_1,z_2,x,y){\rm Tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}$$
$$+ \text{ NLO contribution}$$

High-energy expansion in color dipoles



η - rapidity factorization scale

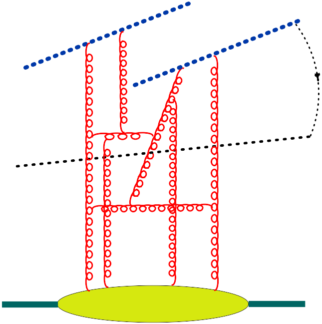
Evolution equation for color dipoles

$$\frac{d}{d\eta}\mathrm{tr}\{U^\eta_x U^{\dagger\eta}_y\} = \frac{\alpha_s}{2\pi^2}\int d^2z \frac{(x-y)^2}{(x-z)^2(y-z)^2}[\mathrm{tr}\{U^\eta_x U^{\dagger\eta}_y\}\mathrm{tr}\{U^\eta_x U^{\dagger\eta}_y\}$$
$$- N_c\mathrm{tr}\{U^\eta_x U^{\dagger\eta}_y\}] + \alpha_s K_{\mathrm{NLO}}\mathrm{tr}\{U^\eta_x U^{\dagger\eta}_y\} + O(\alpha_s^2)$$

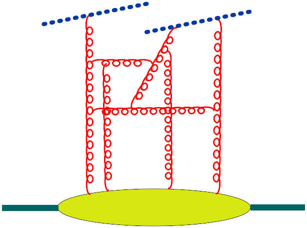
(Linear part of K_{NLO} = $K_{\mathrm{NLO\ BFKL}}$)

Evolution equation for color dipoles

To get the evolution equation, consider the dipole with the rapidies up to η_1 and integrate over the gluons with rapidities $\eta_1 > \eta > \eta_2$. This integral gives the kernel of the evolution equation (multiplied by the dipole(s) with rapidities up to η_2).



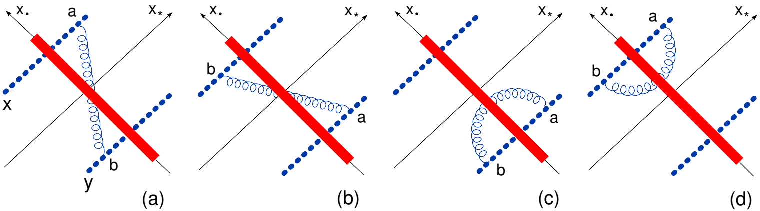
$$\alpha_s(\eta_1 - \eta_2)K_{\text{evol}} \;\otimes$$



Evolution equation in the leading order

$$\frac{d}{d\eta}\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\} = K_{\mathrm{LO}}\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\} + \ldots \quad \Rightarrow$$

$$\frac{d}{d\eta}\langle\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\}\rangle_{\mathrm{shockwave}} = \langle K_{\mathrm{LO}}\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\}\rangle_{\mathrm{shockwave}}$$



$$U_z^{ab} = \mathrm{Tr}\{t^a U_z t^b U_z^\dagger\} \quad \Rightarrow (U_x U_y^\dagger)^{\eta_1} \to (U_x U_y^\dagger)^{\eta_1} + \alpha_s(\eta_1 - \eta_2)(U_x U_z^\dagger U_z U_y^\dagger)^{\eta_2}$$

$\Rightarrow$ Evolution equation is non-linear

Non linear evolution equation

$$\hat{\mathcal{U}}(x,y) \equiv 1 - \frac{1}{N_c}\mathrm{Tr}\{\hat{U}(x_\perp)\hat{U}^\dagger(y_\perp)\}$$

BK equation

$$\frac{d}{d\eta}\hat{\mathcal{U}}(x,y) \;=\; \frac{\alpha_s N_c}{2\pi^2}\int\frac{d^2z\;(x-y)^2}{(x-z)^2(y-z)^2}\Big\{\hat{\mathcal{U}}(x,z)+\hat{\mathcal{U}}(z,y)-\hat{\mathcal{U}}(x,y)-\hat{\mathcal{U}}(x,z)\hat{\mathcal{U}}(z,y)\Big\}$$

I. B. (1996), Yu. Kovchegov (1999)
Alternative approach: JIMWLK (1997-2000)

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LLA for DIS in pQCD $\Rightarrow$ BFKL (LLA: $\alpha_s \ll 1, \alpha_s\eta \sim 1$)

LLA for DIS in sQCD $\Rightarrow$ BK eqn (LLA: $\alpha_s \ll 1, \alpha_s\eta \sim 1$, $\alpha_s A^{1/3} \sim 1$)

(s for semiclassical)

Why NLO correction?

- To check that high-energy OPE works at the NLO level.
- To check conformal invariance of the NLO BK equation(in $\mathcal{N}$=4 SYM)
- To determine the argument of the coupling constant of the BK equation(in QCD).
- To get the region of application of the leading order evolution equation.

Conformal invariance of the BK equation

Formally, a light-like Wilson line

$$[\infty p_1 + x_\perp, -\infty p_1 + x_\perp] \;=\; \mathrm{Pexp}\left\{ig\int_{-\infty}^{\infty} dx^+ \, A_+(x^+, x_\perp)\right\}$$

is invariant under inversion (with respect to the point with $x^- = 0$).

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Indeed,
$(x^+, x_\perp)^2 = -x_\perp^2 \Rightarrow$ after the inversion $x_\perp \to x_\perp / x_\perp^2$ and $x^+ \to x^+ / x_\perp^2$

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$\Rightarrow$The dipole kernel is invariant under the inversion $V(x_\perp) = U(x_\perp/x_\perp^2)$

$$\frac{d}{d\eta}\mathrm{Tr}\{V_x V_y^\dagger\} = \frac{\alpha_s}{2\pi^2}\int \frac{d^2z}{\not z^4}\,\frac{(x-y)^2\;\not z^4}{(x-z)^2(z-y)^2}\big[\mathrm{Tr}\{V_x V_z^\dagger\}\mathrm{Tr}\{V_z V_y^\dagger\} - N_c \mathrm{Tr}\{V_x V_y^\dagger\}\big]$$

Conformal invariance of the BK equation

SL(2,C) for Wilson lines

$$\hat{S}_- \equiv \frac{i}{2}(K^1 + iK^2), \quad \hat{S}_0 \equiv \frac{i}{2}(D + iM^{12}), \quad \hat{S}_+ \equiv \frac{i}{2}(P^1 - iP^2)$$

$$[\hat{S}_0, \hat{S}_\pm] = \pm\hat{S}_\pm, \quad \frac{1}{2}[\hat{S}_+, \hat{S}_-] = \hat{S}_0,$$

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Conformal invariance of the evolution kernel

$$\frac{d}{d\eta}[\hat{S}_-, \mathrm{Tr}\{U_x U_y^\dagger\}] = \frac{\alpha_s N_c}{2\pi^2}\int dz\, K(x,y,z)[\hat{S}_-, \mathrm{Tr}\{U_x U_z^\dagger\}\mathrm{Tr}\{U_z U_y^\dagger\}]$$

$$\Rightarrow \left[x^2\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y} + z^2\frac{\partial}{\partial z}\right]K(x,y,z) = 0$$

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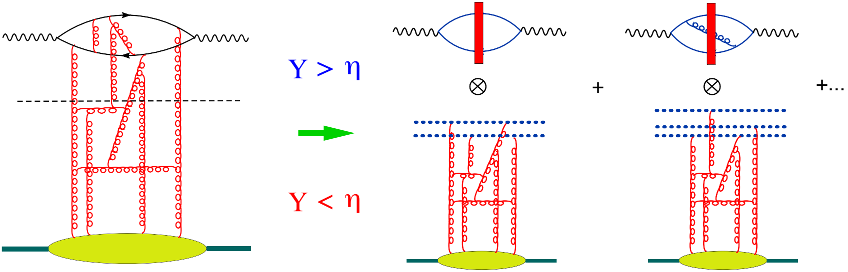
In the leading order - OK. In the NLO - ?



The high-energy operator expansion is $\mathcal{O} \equiv \mathrm{Tr}\{Z^2\}$

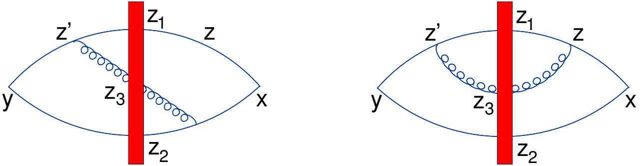
$$T\{\hat{\mathcal{O}}(x)\hat{\mathcal{O}}(y)\} = \int d^2z_1 d^2z_2\ I^{\mathrm{LO}}(z_1,z_2)\mathrm{Tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}$$

$$+\int d^2z_1 d^2z_2 d^2z_3\ I^{\mathrm{NLO}}(z_1,z_2,z_3)[\frac{1}{N_c}\mathrm{Tr}\{T^n\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_3}T^n\hat{U}^\eta_{z_3}\hat{U}^{\dagger\eta}_{z_2}\} - \mathrm{Tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}]$$

In the leading order - conf. invariant impact factor

$$I_{\mathrm{LO}} = \frac{x_+^{-2}y_+^{-2}}{\pi^2\mathcal{Z}_1^2\mathcal{Z}_2^2}, \qquad \mathcal{Z}_i \equiv \frac{(x-z_i)^2_\perp}{x_+} - \frac{(y-z_i)^2_\perp}{y_+} \qquad CCP, 2007$$

NLO impact factor



(a) (b)

$$I^{\mathrm{NLO}}(x,y;z_1,z_2,z_3;\eta) \;=\; -\,I^{\mathrm{LO}} \times \frac{\lambda}{\pi^2}\frac{z_{13}^2}{z_{12}^2 z_{23}^2}\Big[\ln\frac{\sigma s}{4}\mathcal{Z}_3 - \frac{i\pi}{2} + C\Big]$$

The NLO impact factor is not Möbius invariant $\Leftarrow$ the color dipole with the cutoff η is not invariant

However, if we define a composite operator (a - analog of μ^{-2} for usual OPE)

$$[\mathrm{Tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}]^{\mathrm{conf}} \;=\; \mathrm{Tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}$$

$$+\,\frac{\lambda}{2\pi^2}\int d^2 z_3\;\frac{z_{12}^2}{z_{13}^2 z_{23}^2}[\mathrm{Tr}\{T^n\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_3}T^n\hat U^\eta_{z_3}\hat U^{\dagger\eta}_{z_2}\} - N_c\mathrm{Tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}]\ln\frac{a z_{12}^2}{z_{13}^2 z_{23}^2} \;+\; O(\lambda^2)$$

the impact factor becomes conformal in the NLO.

Operator expansion in conformal dipoles

$$T\{\hat{\mathcal{O}}(x)\hat{\mathcal{O}}(y)\} = \int d^2z_1 d^2z_2 \; I^{\rm LO}(z_1,z_2){\rm Tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}^{\rm conf}$$

$$+\int d^2z_1 d^2z_2 d^2z_3 \; I^{\rm NLO}(z_1,z_2,z_3)[\frac{1}{N_c}{\rm Tr}\{T^n\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_3}T^n\hat{U}^\eta_{z_3}\hat{U}^{\dagger\eta}_{z_2}\} - {\rm Tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}]$$

$$I^{\rm NLO} = \; - I^{\rm LO}\frac{\lambda}{2\pi^2}\int dz_3 \frac{z_{12}^2}{z_{13}^2 z_{23}^2}\Big[\ln\frac{z_{12}^2 e^{2\eta}as^2}{z_{13}^2 z_{23}^2}\mathcal{Z}_3^2 - i\pi + 2C\Big]$$

The new NLO impact factor is conformally invariant
$\Rightarrow$ ${\rm Tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}^{\rm conf}$ is Möbius invariant

We think that one can construct the composite conformal dipole operator order by order in perturbation theory.

Analogy: when the UV cutoff does not respect the symmetry of a local operator, the composite local renormalized operator in must be corrected by finite counterterms order by order in perturbaton theory.

Definition of the NLO kernel

In general

$$\frac{d}{d\eta}\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\} = \alpha_s K_{\mathrm{LO}}\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\} + \alpha_s^2 K_{\mathrm{NLO}}\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\} + O(\alpha_s^3)$$

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$$\alpha_s^2 K_{\mathrm{NLO}}\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\} = \frac{d}{d\eta}\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\} - \alpha_s K_{\mathrm{LO}}\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\} + O(\alpha_s^3)$$

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We calculate the "matrix element" of the r.h.s. in the shock-wave background

$$\langle\alpha_s^2 K_{\mathrm{NLO}}\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\}\rangle = \frac{d}{d\eta}\langle\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\}\rangle - \langle\alpha_s K_{\mathrm{LO}}\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\}\rangle + O(\alpha_s^3)$$

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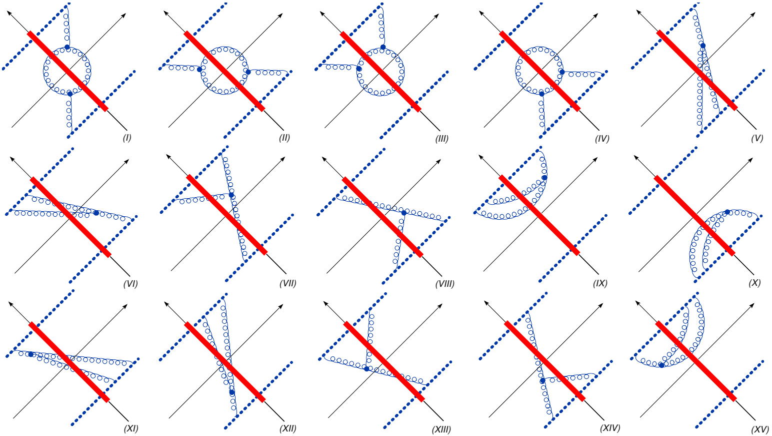
Subtraction of the (LO) contribution (with the rigid rapidity cutoff)

$\Rightarrow$ $\left[\frac{1}{v}\right]_+$ prescription in the integrals over Feynman parameter v

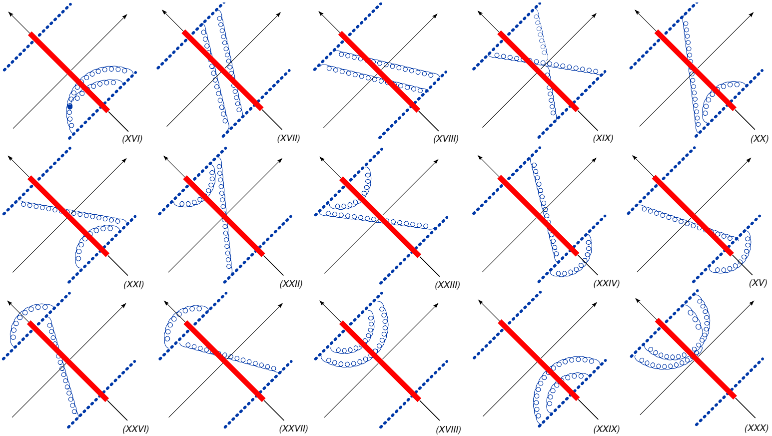
Typical integral

$$\int_0^1 dv\,\frac{1}{(k-p)_\perp^2 v + p_\perp^2(1-v)}\left[\frac{1}{v}\right]_+ = \frac{1}{p_\perp^2}\ln\frac{(k-p)_\perp^2}{p_\perp^2}$$

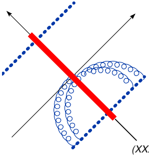
Gluon part of the NLO BK kernel: diagrams

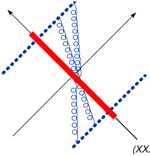


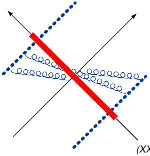
Diagrams for 1→3 dipoles transition

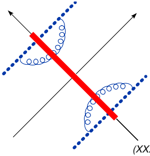


Diagrams for 1→3 dipoles transition

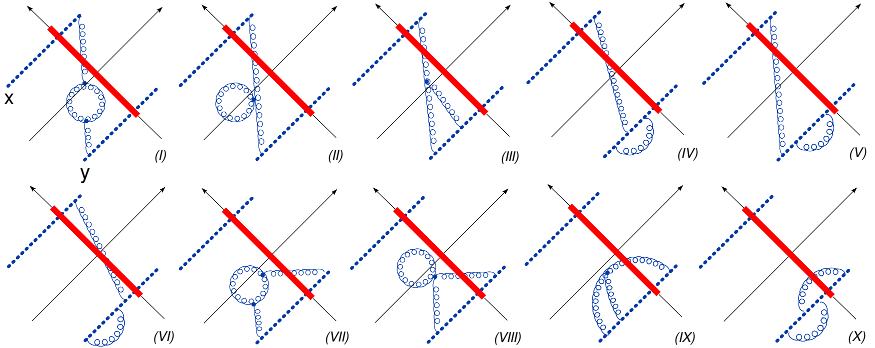

(XXXI)


(XXXII)

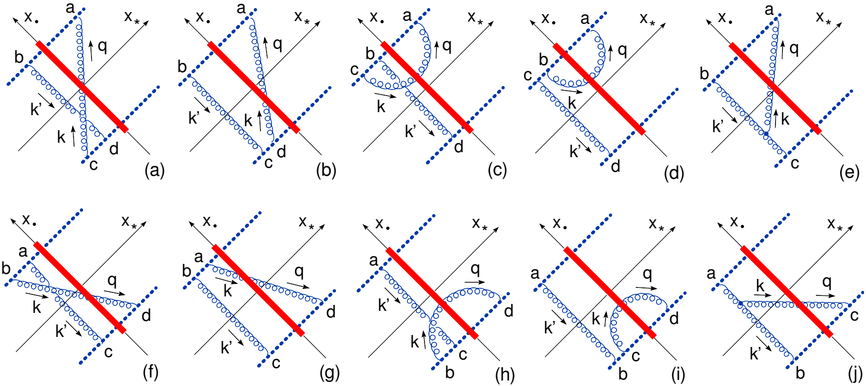

(XXXIII)


(XXXIV)

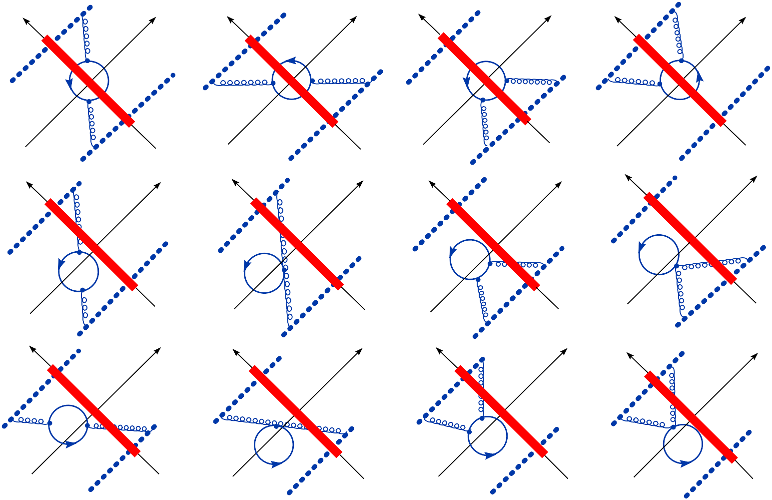
"Running coupling" diagrams



$1 \rightarrow 2$ dipole transition diagrams



Gluino and scalar loops



Evolution equation for color dipole in $\mathcal{N}=4$ (I.B. and G. Chirilli)

$$
\begin{aligned}
&\frac{d}{d\eta}\mathrm{Tr}\{\hat{U}^{\eta}_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}\\
&= \frac{\alpha_s}{\pi^2}\int d^2z_3\ \frac{z_{12}^2}{z_{13}^2z_{23}^2}\Big\{1-\frac{\alpha_sN_c}{4\pi}\Big[\frac{\pi^2}{3}+2\ln\frac{z_{13}^2}{z_{12}^2}\ln\frac{z_{23}^2}{z_{12}^2}\Big]\Big\}\\
&\times\ [\mathrm{Tr}\{T^a\hat{U}^{\eta}_{z_1}\hat{U}^{\dagger\eta}_{z_3}T^a\hat{U}^{\eta}_{z_3}\hat{U}^{\dagger\eta}_{z_2}\}-N_c\mathrm{Tr}\{\hat{U}^{\eta}_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}]\\
&-\frac{\alpha_s^2}{4\pi^4}\int\frac{d^2z_3d^2z_4}{z_{34}^4}\ \frac{z_{12}^2z_{34}^2}{z_{13}^2z_{24}^2}\Big[1+\frac{z_{12}^2z_{34}^2}{z_{13}^2z_{24}^2-z_{23}^2z_{14}^2}\Big]\ln\frac{z_{13}^2z_{24}^2}{z_{14}^2z_{23}^2}\\
&\times\ \mathrm{Tr}\{[T^a,T^b]\hat{U}^{\eta}_{z_1}T^{a'}T^{b'}\hat{U}^{\dagger\eta}_{z_2}+T^bT^a\hat{U}^{\eta}_{z_1}[T^{b'},T^{a'}]\hat{U}^{\dagger\eta}_{z_2}\}(\hat{U}^{\eta}_{z_3})^{aa'}(\hat{U}^{\eta}_{z_4}-\hat{U}^{\eta}_{z_3})^{bb'}
\end{aligned}
$$

NLO kernel = Non-conformal term + Conformal term.

Non-conformal term is due to the non-invariant cutoff $\alpha<\sigma=e^{2\eta}$ in the rapidity of Wilson lines.

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&-\frac{\alpha_s^2}{4\pi^4}\int\frac{d^2z_3d^2z_4}{z_{34}^4}\ \frac{z_{12}^2z_{34}^2}{z_{13}^2z_{24}^2}\Big[1+\frac{z_{12}^2z_{34}^2}{z_{13}^2z_{24}^2-z_{23}^2z_{14}^2}\Big]\ln\frac{z_{13}^2z_{24}^2}{z_{14}^2z_{23}^2}\\
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\end{aligned}
$$

NLO kernel = Non-conformal term + Conformal term.

Non-conformal term is due to the non-invariant cutoff $\alpha<\sigma=e^{2\eta}$ in the rapidity of Wilson lines.

For the conformal composite dipole the result is Möbius invariant

Evolution equation for composite conformal dipoles in $\mathcal{N}=4$

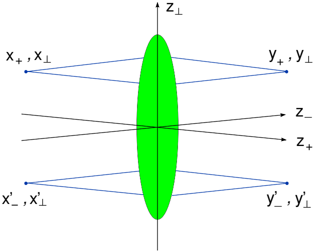
$$
\begin{aligned}
&\frac{d}{d\eta}\big[\mathrm{Tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}\big]^{\rm conf}\\
&= \frac{\alpha_s}{\pi^2}\int d^2z_3\ \frac{z_{12}^2}{z_{13}^2z_{23}^2}\Big[1-\frac{\alpha_sN_c}{4\pi}\frac{\pi^2}{3}\Big]\big[\mathrm{Tr}\{T^a\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_3}T^a\hat U_{z_3}\hat U^{\dagger\eta}_{z_2}\}-N_c\mathrm{Tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}\big]^{\rm conf}\\
&\quad-\frac{\alpha_s^2}{4\pi^4}\int d^2z_3d^2z_4\frac{z_{12}^2}{z_{13}^2z_{24}^2z_{34}^2}\Big\{2\ln\frac{z_{12}^2z_{34}^2}{z_{14}^2z_{23}^2}+\Big[1+\frac{z_{12}^2z_{34}^2}{z_{13}^2z_{24}^2-z_{14}^2z_{23}^2}\Big]\ln\frac{z_{13}^2z_{24}^2}{z_{14}^2z_{23}^2}\Big\}\\
&\quad\times\mathrm{Tr}\{[T^a,T^b]\hat U^\eta_{z_1}T^{a'}T^{b'}\hat U^{\dagger\eta}_{z_2}+T^bT^a\hat U^\eta_{z_1}[T^{b'},T^{a'}]\hat U^{\dagger\eta}_{z_2}\}[(\hat U^\eta_{z_3})^{aa'}(\hat U^\eta_{z_4})^{bb'}-(z_4\to z_3)]
\end{aligned}
$$

Now Möbius invariant!

Small-x (Regge) limit in the coordinate space

$$(x-y)^4(x'-y')^4\langle\mathcal{O}(x)\mathcal{O}^\dagger(y)\mathcal{O}(x')\mathcal{O}^\dagger(y')\rangle$$

Regge limit: $x_+ \to \rho x_+,\ x'_+ \to \rho x'_+,\ y_- \to \rho' y_-,\ y'_- \to \rho' y-_-$ $\quad \rho, \rho' \to \infty$



Regge limit symmetry in a conformal theory: 2-dim conformal Möbius group $SL(2,C)$.

Small-x (Regge) limit in the coordinate space

$$(x-y)^4(x'-y')^4\langle\mathcal{O}(x)\mathcal{O}^\dagger(y)\mathcal{O}(x')\mathcal{O}^\dagger(y')\rangle$$

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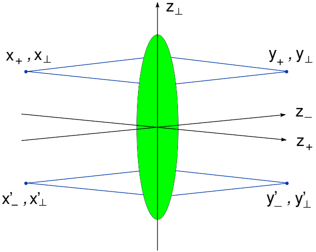

LLA: $\alpha_s \ll 1,\ \alpha_s \ln\rho \sim 1,\ \Rightarrow \sum(\alpha_s\ln\rho)^n \equiv$ BFKL pomeron.
LLA $\Leftrightarrow$ tree diagrams $\Rightarrow$ the BFKL pomeron is Möbius invariant .

NLO LLA: extra α_s: $\sum\alpha_s(\alpha_s\ln\rho)^n \equiv$ NLO BFKL

In conformal theory ($\mathcal{N}=4$ SYM) the NLO BFKL for composite conformal dipole operator is Möbius invariant.

NLO Amplitude in $\mathcal{N}$=4 SYM theory

The pomeron contribution to a 4-point correlation function in $\mathcal{N} = 4$ SYM can be represented as $\lambda \equiv g^2 N_c$

$$(x-y)^4(x'-y')^4\langle \mathcal{O}(x)\mathcal{O}^\dagger(y)\mathcal{O}(x')\mathcal{O}^\dagger(y')\rangle$$
$$= \frac{i}{8\pi^2}\int d\nu\, \tilde{f}_+(\nu)\tanh \pi\nu \frac{\sin \nu\rho}{\sinh \rho} F(\nu,\lambda) R^{\frac{1}{2}\omega(\nu,\lambda)} \qquad \text{Cornalba(2007)}$$

$\omega(\nu,\lambda) = \frac{\lambda}{\pi}\chi(\nu) + \lambda^2\omega_1(\nu) + ...$ is the pomeron intercept,
$\chi(\nu) = 2\psi(1) - \psi(\gamma) - \psi(1-\gamma), \quad \gamma \equiv \frac{1}{2} + i\nu$
$\tilde{f}_+(\omega) = (e^{i\pi\omega} - 1)/\sin \pi\omega$ is the signature factor.

$F(\nu,\lambda) = F_0(\nu) + \lambda F_1(\nu) + ...$ is the "pomeron residue".

R and r are two conformal ratios:

$$R = \frac{(x-x')(y-y')^2}{(x-y)^2(x'-y')^2}, \quad r = R\Big[1 - \frac{(x-y')^2(y-x')^2}{(x-x')^2(y-y')^2} + \frac{1}{R}\Big]^2, \quad \cosh\rho = \frac{\sqrt{r}}{2}$$

In the Regge limit $s \to \infty$ the ratio R scales as s while r does not depend on energy.

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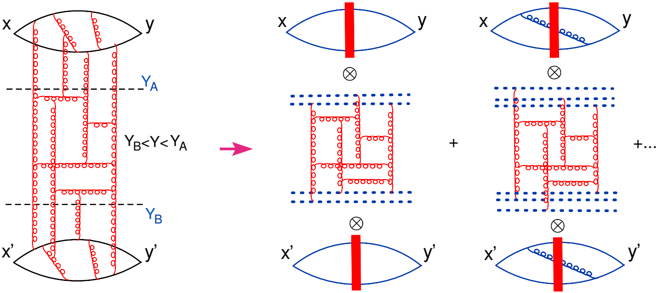
In the Regge limit $s \to \infty$ the ratio R scales as s while r does not depend on energy.

We reproduced $\omega_1(\nu)$ (Lipatov & Kotikov, 2000) and found $F_1(\nu)$

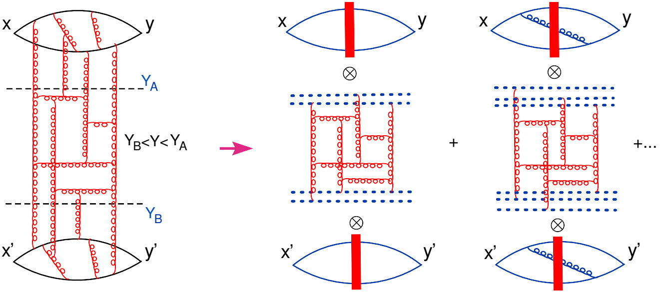
NLO Amplitude in $\mathcal{N}$=4 SYM theory: factorization in rapidity



$$(x-y)^4(x'-y')^4\langle T\{\hat{\mathcal{O}}(x)\hat{\mathcal{O}}^\dagger(y)\hat{\mathcal{O}}(x')\hat{\mathcal{O}}^\dagger(y')\}\rangle$$
$$= \int d^2z_{1\perp}d^2z_{2\perp}d^2z'_{1\perp}d^2z'_{2\perp}\,\mathrm{IF}^{a_0}(x,y;z_1,z_2)[\mathrm{DD}]^{a_0,b_0}(z_1,z_2;z'_1,z'_2)\mathrm{IF}^{b_0}(x',y';z'_1,z'_2)$$

$a_0 = \frac{x_+y_+}{(x-y)^2}$, $b_0 = \frac{x'_-y'_-}{(x'-y')^2}$ $\Leftrightarrow$ impact factors do not scale with energy

$\Rightarrow$ all energy dependence is contained in $[\mathrm{DD}]^{a_0,b_0}$ $(a_0b_0 = R)$

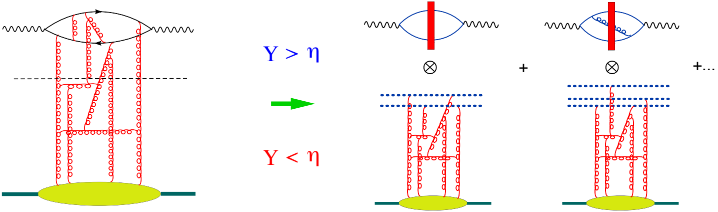


$$(x-y)^4(x'-y')^4\langle T\{\hat{\mathcal{O}}(x)\hat{\mathcal{O}}^\dagger(y)\hat{\mathcal{O}}(x')\hat{\mathcal{O}}^\dagger(y')\}\rangle$$

$$= \int d^2z_{1\perp}d^2z_{2\perp}d^2z'_{1\perp}d^2z'_{2\perp}\mathrm{IF}^{a_0}(x,y;z_1,z_2)[\mathrm{DD}]^{a_0,b_0}(z_1,z_2;z'_1,z'_2)\mathrm{IF}^{b_0}(x',y';z'_1,z'_2)$$

Result : (G.A. Chirilli and I.B.)

$$F(\nu) = \frac{N_c^2}{N_c^2-1}\frac{4\pi^4\alpha_s^2}{\cosh^2\pi\nu}\left\{1+\frac{\alpha_sN_c}{\pi}\left[-\frac{2\pi^2}{\cosh^2\pi\nu}+\frac{\pi^2}{2}-\frac{8}{1+4\nu^2}\right]+O(\alpha_s^2)\right\}$$



DIS structure function $F_2(x)$: photon impact factor + evolution of color dipoles+ initial conditions for the small-x evolution

Photon impact factor in the LO

$$(x-y)^4 T\{\hat{\bar\psi}(x)\gamma^\mu\hat\psi(x)\hat{\bar\psi}(y)\gamma^\nu\hat\psi(y)\} = \int \frac{d^2z_1 d^2z_2}{z_{12}^4}\, I^{\rm LO}_{\mu\nu}(z_1,z_2)\mathrm{tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}$$

$$I^{\rm LO}_{\mu\nu}(z_1,z_2) = \frac{\mathcal{R}^2}{\pi^6(\kappa\cdot\zeta_1)(\kappa\cdot\zeta_2)}\frac{\partial^2}{\partial x^\mu\partial y^\nu}\Big[(\kappa\cdot\zeta_1)(\kappa\cdot\zeta_2)-\frac{1}{2}\kappa^2(\zeta_1\cdot\zeta_2)\Big].$$

$$\kappa \equiv \frac{1}{\sqrt{s}x^+}\Big(\frac{p_1}{s}-x^2p_2+x_\perp\Big)-\frac{1}{\sqrt{s}y^+}\Big(\frac{p_1}{s}-y^2p_2+y_\perp\Big)$$

$$\zeta_i \equiv \Big(\frac{p_1}{s}+z_{i\perp}^2p_2+z_{i\perp}\Big), \qquad \mathcal{R} \equiv \frac{\kappa^2(\zeta_1\cdot\zeta_2)}{2(\kappa\cdot\zeta_1)(\kappa\cdot\zeta_2)}$$

Composite "conformal" dipole $[\mathrm{tr}\{\hat{U}_{z_1}\hat{U}^\dagger_{z_2}\}]_{a_0}$ - same as in $\mathcal{N}=4$ case.

$$(x-y)^4 T\{\bar{\hat{\psi}}(x)\gamma^\mu\hat{\psi}(x)\bar{\hat{\psi}}(y)\gamma^\nu\hat{\psi}(y)\}$$

$$= \int\frac{d^2z_1d^2z_2}{z_{12}^4}\Big\{I^{\mu\nu}_{\rm LO}(z_1,z_2)\big[1+\frac{\alpha_s}{\pi}\big][\mathrm{tr}\{\hat{U}_{z_1}\hat{U}^\dagger_{z_2}\}]_{a_0}$$

$$+\int d^2z_3\Big[\frac{\alpha_s}{4\pi^2}\frac{z_{12}^2}{z_{13}^2z_{23}^2}\Big(\ln\frac{\kappa^2(\zeta_1\cdot\zeta_3)(\zeta_1\cdot\zeta_3)}{2(\kappa\cdot\zeta_3)^2(\zeta_1\cdot\zeta_2)}-2C\Big)I^{\mu\nu}_{\rm LO}+\ I^{\mu\nu}_2\Big]$$

$$\times\ [\mathrm{tr}\{\hat{U}_{z_1}\hat{U}^\dagger_{z_3}\}\mathrm{tr}\{\hat{U}_{z_3}\hat{U}^\dagger_{z_2}\}-N_c\mathrm{tr}\{\hat{U}_{z_1}\hat{U}^\dagger_{z_2}\}]_{a_0}\Big\}$$

$$(I_2)_{\mu\nu}(z_1,z_2,z_3)\ =\ \frac{\alpha_s}{16\pi^8}\frac{\mathcal{R}^2}{(\kappa\cdot\zeta_1)(\kappa\cdot\zeta_2)}\Big\{\frac{(\kappa\cdot\zeta_2)}{(\kappa\cdot\zeta_3)}\frac{\partial^2}{\partial x^\mu\partial y^\nu}\Big[-\frac{(\kappa\cdot\zeta_1)^2}{(\zeta_1\cdot\zeta_3)}$$

$$+\frac{(\kappa\cdot\zeta_1)(\kappa\cdot\zeta_2)}{(\zeta_2\cdot\zeta_3)}+\ \frac{(\kappa\cdot\zeta_1)(\kappa\cdot\zeta_3)(\zeta_1\cdot\zeta_2)}{(\zeta_1\cdot\zeta_3)(\zeta_2\cdot\zeta_3)}-\frac{\kappa^2(\zeta_1\cdot\zeta_2)}{(\zeta_2\cdot\zeta_3)}\Big]$$

$$+\frac{(\kappa\cdot\zeta_2)^2}{(\kappa\cdot\zeta_3)^2}\frac{\partial^2}{\partial x^\mu\partial y^\nu}\Big[\frac{(\kappa\cdot\zeta_1)(\kappa\cdot\zeta_3)}{(\zeta_2\cdot\zeta_3)}-\frac{\kappa^2(\zeta_1\cdot\zeta_3)}{2(\zeta_2\cdot\zeta_3)}\Big]+(\zeta_1\leftrightarrow\zeta_2)\Big\}$$

With two-gluon (NLO BFKL) accuracy

$$\frac{1}{N_c}(x-y)^4 T\{\bar{\hat{\psi}}(x)\gamma^\mu\hat{\psi}(x)\bar{\hat{\psi}}(y)\gamma^\nu\hat{\psi}(y)\} = \frac{\partial\kappa^\alpha}{\partial x^\mu}\frac{\partial\kappa^\beta}{\partial y^\nu}\int\frac{dz_1 dz_2}{z_{12}^4}\,\hat{\mathcal{U}}_{a_0}(z_1,z_2)\left[\mathcal{I}^{\rm LO}_{\alpha\beta}\left(1+\frac{\alpha_s}{\pi}\right)+\mathcal{I}^{\rm NLO}_{\alpha\beta}\right]$$

$$\mathcal{I}^{\alpha\beta}_{\rm LO}(x,y;z_1,z_2) = \mathcal{R}^2\frac{g^{\alpha\beta}(\zeta_1\cdot\zeta_2)-\zeta_1^\alpha\zeta_2^\beta-\zeta_2^\alpha\zeta_1^\beta}{\pi^6(\kappa\cdot\zeta_1)(\kappa\cdot\zeta_2)}$$

$$\begin{aligned}\mathcal{I}^{\alpha\beta}_{\rm NLO}(x,y;z_1,z_2) = {}& \frac{\alpha_s N_c}{4\pi^7}\mathcal{R}^2\Big\{\frac{\zeta_1^\alpha\zeta_2^\beta+\zeta_1\leftrightarrow\zeta_2}{(\kappa\cdot\zeta_1)(\kappa\cdot\zeta_2)}\Big[4{\rm Li}_2(1-\mathcal{R})-\frac{2\pi^2}{3}+\frac{2\ln\mathcal{R}}{1-\mathcal{R}}+\frac{\ln\mathcal{R}}{\mathcal{R}} \\ &-4\ln\mathcal{R}+\frac{1}{2\mathcal{R}}-2+2\Big(\ln\frac{1}{\mathcal{R}}+\frac{1}{\mathcal{R}}-2\Big)\Big(\ln\frac{1}{\mathcal{R}}+2C\Big)-4C-\frac{2C}{\mathcal{R}}\Big] \\ &+\Big(\frac{\zeta_1^\alpha\zeta_1^\beta}{(\kappa\cdot\zeta_1)^2}+\zeta_1\leftrightarrow\zeta_2\Big)\Big[\frac{\ln\mathcal{R}}{\mathcal{R}}-\frac{2C}{\mathcal{R}}+2\frac{\ln\mathcal{R}}{1-\mathcal{R}}-\frac{1}{2\mathcal{R}}\Big]-\frac{2}{\kappa^2}\Big(g^{\alpha\beta}-2\frac{\kappa^\alpha\kappa^\beta}{\kappa^2}\Big) \\ &+\Big[\frac{\zeta_1^\alpha\kappa^\beta+\zeta_1^\beta\kappa^\alpha}{(\kappa\cdot\zeta_1)\kappa^2}+\zeta_1\leftrightarrow\zeta_2\Big]\Big[-2\frac{\ln\mathcal{R}}{1-\mathcal{R}}-\frac{\ln\mathcal{R}}{\mathcal{R}}+\ln\mathcal{R}-\frac{3}{2\mathcal{R}}+\frac{5}{2}+2C+\frac{2C}{\mathcal{R}}\Big] \\ &+\frac{g^{\alpha\beta}(\zeta_1\cdot\zeta_2)}{(\kappa\cdot\zeta_1)(\kappa\cdot\zeta_2)}\Big[\frac{2\pi^2}{3}-4{\rm Li}_2(1-\mathcal{R}) \\ &-2\Big(\ln\frac{1}{\mathcal{R}}+\frac{1}{\mathcal{R}}+\frac{1}{2\mathcal{R}^2}-3\Big)\Big(\ln\frac{1}{\mathcal{R}}+2C\Big)+6\ln\mathcal{R}-\frac{2}{\mathcal{R}}+2+\frac{3}{2\mathcal{R}^2}\Big]\Big\}\end{aligned}$$

5 tensor structures (CCP, 2009)

NLO impact factor for DIS

$$I^{\mu\nu}(q,k_\perp) = \frac{N_c}{32}\int\frac{d\nu}{\pi\nu}\frac{\sinh\pi\nu}{(1+\nu^2)\cosh^2\pi\nu}\Big(\frac{k_\perp^2}{Q^2}\Big)^{\frac{1}{2}-i\nu}$$
$$\times\Big\{\Big[\Big(\frac{9}{4}+\nu^2\Big)\Big(1+\frac{\alpha_s}{\pi}+\frac{\alpha_s N_c}{2\pi}\mathcal{F}_1(\nu)\Big)P_1^{\mu\nu} + \Big(\frac{11}{4}+3\nu^2\Big)\Big(1+\frac{\alpha_s}{\pi}+\frac{\alpha_s N_c}{2\pi}\mathcal{F}_2(\nu)\Big)P_2^{\mu\nu}\Big]$$

$$P_1^{\mu\nu} = g^{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \qquad P_2^{\mu\nu} = \frac{1}{q^2}\Big(q^\mu - \frac{p_2^\mu q^2}{q\cdot p_2}\Big)\Big(q^\nu - \frac{p_2^\nu q^2}{q\cdot p_2}\Big)$$

$$\mathcal{F}_{1(2)}(\nu) = \Phi_{1(2)}(\nu) + \chi_\gamma\Psi(\nu),$$
$$\Psi(\nu) \equiv \psi(\bar\gamma) + 2\psi(2-\gamma) - 2\psi(4-2\gamma) - \psi(2+\gamma), \qquad \gamma \equiv \frac{1}{2}+i\nu$$

$$\Phi_1(\nu) = F(\gamma) + \frac{3\chi_\gamma}{2+\bar\gamma\gamma} + 1 + \frac{25}{18(2-\gamma)} + \frac{1}{2\bar\gamma} - \frac{1}{2\gamma} - \frac{7}{18(1+\gamma)} + \frac{10}{3(1+\gamma)^2}$$
$$\Phi_2(\nu) = F(\gamma) + \frac{3\chi_\gamma}{2+\bar\gamma\gamma} + 1 + \frac{1}{2\bar\gamma\gamma} - \frac{7}{2(2+3\bar\gamma\gamma)} + \frac{\chi_\gamma}{1+\gamma} + \frac{\chi_\gamma(1+3\gamma)}{2+3\bar\gamma\gamma}$$

$$F(\gamma) = \frac{2\pi^2}{3} - \frac{2\pi^2}{\sin^2\pi\gamma} - 2C\chi_\gamma + \frac{\chi_\gamma - 2}{\bar\gamma\gamma}$$

NLO evolution of composite “conformal” dipoles in QCD

I. B. and G. Chirilli

$$a\frac{d}{da}[\mathrm{tr}\{U_{z_1}U^\dagger_{z_2}\}]_a^{\rm comp} = \frac{\alpha_s}{2\pi^2}\int d^2z_3\,\Big([\mathrm{tr}\{U_{z_1}U^\dagger_{z_3}\}\mathrm{tr}\{U_{z_3}U^\dagger_{z_2}\} - N_c\mathrm{tr}\{U_{z_1}U^\dagger_{z_2}\}]_a^{\rm comp}$$

$$\times\frac{z_{12}^2}{z_{13}^2z_{23}^2}\Big[1+\frac{\alpha_sN_c}{4\pi}\Big(b\ln z_{12}^2\mu^2 + b\frac{z_{13}^2-z_{23}^2}{z_{13}^2z_{23}^2}\ln\frac{z_{13}^2}{z_{23}^2}+\frac{67}{9}-\frac{\pi^2}{3}\Big)\Big]$$

$$+\frac{\alpha_s}{4\pi^2}\int\frac{d^2z_4}{z_{34}^4}\Big\{\Big[-2+\frac{z_{23}{}^2z_{23}^2+z_{24}^2z_{13}^2-4z_{12}^2z_{34}^2}{2(z_{23}{}^2z_{23}^2-z_{24}^2z_{13}^2)}\ln\frac{z_{23}{}^2z_{23}^2}{z_{24}^2z_{13}^2}\Big]$$

$$\times\,[\mathrm{tr}\{U_{z_1}U^\dagger_{z_3}\}\mathrm{tr}\{U_{z_3}U^\dagger_{z_4}\}\{U_{z_4}U^\dagger_{z_2}\} - \mathrm{tr}\{U_{z_1}U^\dagger_{z_3}U_{z_4}U^\dagger_{z_2}U_{z_3}U^\dagger_{z_4}\} - (z_4\to z_3)]$$

$$+\frac{z_{12}^2z_{34}^2}{z_{13}^2z_{24}^2}\Big[2\ln\frac{z_{12}^2z_{34}^2}{z_{23}{}^2z_{23}^2}+\Big(1+\frac{z_{12}^2z_{34}^2}{z_{13}^2z_{24}^2-z_{23}{}^2z_{23}^2}\Big)\ln\frac{z_{13}^2z_{24}^2}{z_{23}{}^2z_{23}^2}\Big]$$

$$\times\,[\mathrm{tr}\{U_{z_1}U^\dagger_{z_3}\}\mathrm{tr}\{U_{z_3}U^\dagger_{z_4}\}\mathrm{tr}\{U_{z_4}U^\dagger_{z_2}\} - \mathrm{tr}\{U_{z_1}U^\dagger_{z_4}U_{z_3}U^\dagger_{z_2}U_{z_4}U^\dagger_{z_3}\} - (z_4\to z_3)]\Big\}$$

$$b=\frac{11}{3}N_c-\frac{2}{3}n_f$$

$K_{\rm NLO\ BK}$ = Running coupling part + Conformal "non-analytic" (in j) part + Conformal analytic ($\mathcal{N}=4$) part

Linearized $K_{\rm NLO\ BK}$ reproduces the known result for the forward NLO BFKL kernel.

Argument of coupling constant

$$\frac{d}{d\eta}\hat{\mathcal{U}}(z_1,z_2) =$$

$$\frac{\alpha_s(?_\perp)N_c}{2\pi^2}\int dz_3\,\frac{z_{12}^2}{z_{13}^2 z_{23}^2}\Big\{\hat{\mathcal{U}}(z_1,z_3)+\hat{\mathcal{U}}(z_3,z_2)-\hat{\mathcal{U}}(z_1,z_2)-\hat{\mathcal{U}}(z_1,z_3)\hat{\mathcal{U}}(z_3,z_2)\Big\}$$

Argument of coupling constant

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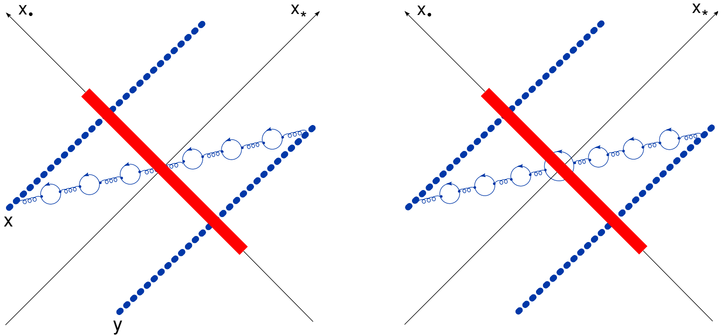
Renormalon-based approach: summation of quark bubbles



$$-\tfrac{2}{3}n_f \rightarrow b = \tfrac{11}{3}N_c - \tfrac{2}{3}n_f$$

Argument of coupling constant (rcBK)

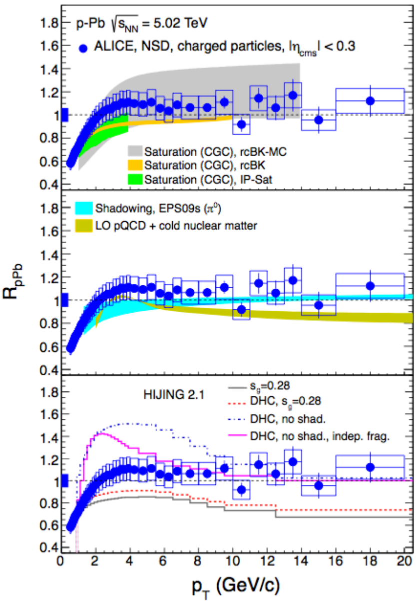
$$\frac{d}{d\eta}\mathrm{Tr}\{\hat{U}_{z_1}\hat{U}^\dagger_{z_2}\} \;=\; \frac{\alpha_s(z_{12}^2)}{2\pi^2}\int d^2z\,\left[\mathrm{Tr}\{\hat{U}_{z_1}\hat{U}^\dagger_{z_3}\}\mathrm{Tr}\{\hat{U}_{z_3}\hat{U}^\dagger_{z_2}\} - N_c\mathrm{Tr}\{\hat{U}_{z_1}\hat{U}^\dagger_{z_2}\}\right]$$
$$\times\left[\frac{z_{12}^2}{z_{13}^2 z_{23}^2} + \frac{1}{z_{13}^2}\left(\frac{\alpha_s(z_{13}^2)}{\alpha_s(z_{23}^2)} - 1\right) + \frac{1}{z_{23}^2}\left(\frac{\alpha_s(z_{23}^2)}{\alpha_s(z_{13}^2)} - 1\right)\right] + \ldots$$

I.B.; Yu. Kovchegov and H. Weigert (2006)

When the sizes of the dipoles are very different the kernel reduces to:

$$\frac{\alpha_s(z_{12}^2)}{2\pi^2}\frac{z_{12}^2}{z_{13}^2 z_{23}^2} \qquad |z_{12}| \ll |z_{13}|, |z_{23}|$$
$$\frac{\alpha_s(z_{13}^2)}{2\pi^2 z_{13}^2} \qquad |z_{13}| \ll |z_{12}|, |z_{23}|$$
$$\frac{\alpha_s(z_{23}^2)}{2\pi^2 z_{23}^2} \qquad |z_{23}| \ll |z_{12}|, |z_{13}|$$

$\Rightarrow$ the argument of the coupling constant is given by the size of the smallest dipole.



ALICE arXiv:1210.4520

Nuclear modification factor

$$R^{pPb}(p_T) = \frac{d^2N^{pPb}_{\rm ch}/d\eta dp_T}{\langle T_{pPb}\rangle d^2\sigma^{\rm pp}_{\rm ch}/d\eta dp_T}$$

N^{pPb} $\equiv$ charged particle yield in p-Pb collisions.

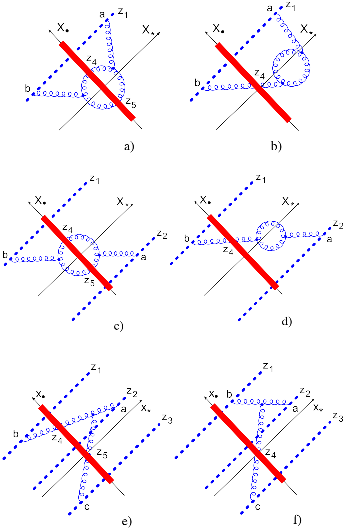
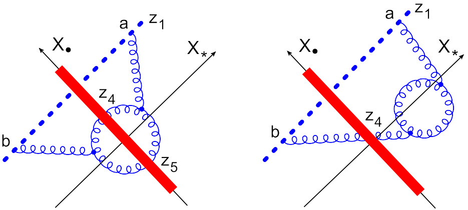


Figure: Typical NLO diagrams: self-interaction (a,b), pairwise interactions (c,d), and triple interaction (e,f)

Self-interaction (gluon reggeization)



a) b)

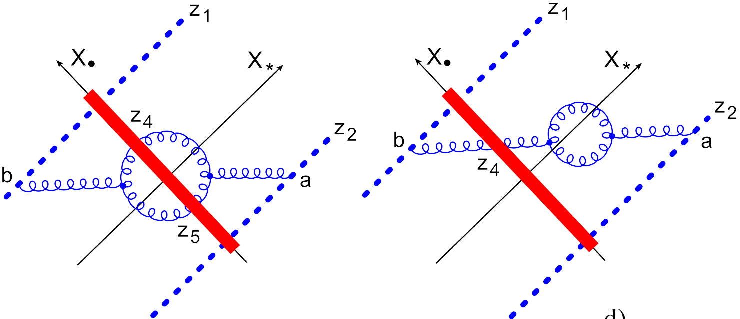
$$\frac{d}{d\eta}(U_1)_{ij} = \frac{\alpha_s^2}{8\pi^4}\int\frac{d^2z_4d^2z_5}{z_{45}^2}\,\Big\{U_4^{dd'}(U_5^{ee'} - U_4^{ee'})$$

$$\times\,\Big(\Big[2I_1 - \frac{4}{z_{45}^2}\Big]f^{ade}f^{bd'e'}(t^aU_1t^b)_{ij} + \frac{(z_{14},z_{15})}{z_{14}^2z_{15}^2}\ln\frac{z_{14}^2}{z_{15}^2}\big[if^{ad'e'}(\{t^d,t^e\}U_1t^a)_{ij} - if^{ade}(t^aU_1\{t^{d'},t^{e'}\})_{ij}\big]\Big)\Big\}$$

$$+\,\frac{\alpha_s^2N_c}{4\pi^3}\int\frac{d^2z_4}{z_{14}^2}\,(U_4^{ab} - U_1^{ab})(t^aU_1t^b)_{ij}\Big\{\Big[\frac{11}{3}\ln z_{14}^2\mu^2 + \frac{67}{9} - \frac{\pi^2}{3}\Big]$$

$$I_1 \;\equiv\; I(z_1,z_4,z_5) \;=\; \frac{\ln z_{14}^2/z_{15}^2}{z_{14}^2 - z_{15}^2}\Big[\frac{z_{14}^2+z_{15}^2}{z_{45}^2} - \frac{(z_{14},z_{15})}{z_{14}^2} - \frac{(z_{14},z_{15})}{z_{15}^2} - 2\Big]$$

Pairwise interaction



c) d)

$$\frac{d}{d\eta}(U_1)_{ij}(U_2)_{kl} \;=\; \frac{\alpha_s^2}{8\pi^4}\int d^2z_4 d^2z_5(\mathcal{A}_1+\mathcal{A}_2+\mathcal{A}_3)+\;\frac{\alpha_s^2}{8\pi^3}\int d^2z_4(\mathcal{B}_1+N_c\mathcal{B}_2)$$

Pairwise interaction

$$\mathcal{A}_1 \;=\; \Big[(t^aU_1)_{ij}(U_2t^b)_{kl}+(U_1t^b)_{ij}(t^aU_2)_{kl}\Big]$$
$$\times\,\Big[f^{ade}f^{bd'e'}U_4^{dd'}(U_5^{ee'}-U_4^{ee'})\Big(-K-\frac{4}{z_{45}^4}+\frac{I_1}{z_{45}^2}+\frac{I_2}{z_{45}^2}\Big)\Big]$$

K = NLO BK kernel for $\mathcal{N}=4$ SYM

$$\mathcal{A}_2 \;=\; 4(U_4-U_1)^{dd'}(U_5-U_2)^{ee'}$$
$$\Big\{i\big[f^{ad'e'}(t^dU_1t^a)_{ij}(t^eU_2)_{kl}-\,f^{ade}(t^aU_1t^{d'})_{ij}(U_2t^{e'})_{kl}\big]J_{1245}\ln\frac{z_{14}^2}{z_{15}^2}$$
$$+\,i\big[f^{ad'e'}(t^dU_1)_{ij}(t^eU_2t^a)_{kl}-f^{ade}(U_1t^{d'})_{ij}(t^aU_2t^{e'})_{kl}\big]J_{2154}\ln\frac{z_{24}^2}{z_{25}^2}\Big\}$$

$$J_{1245}\;\equiv\;J(z_1,z_2,z_4,z_5)\;=\;\frac{(z_{14},z_{25})}{z_{14}^2z_{25}^2z_{45}^2}-2\frac{(z_{15},z_{45})(z_{15},z_{25})}{z_{14}^2z_{15}^2z_{25}^2z_{45}^2}+2\frac{(z_{25},z_{45})}{z_{14}^2z_{25}^2z_{45}^2}$$

Pairwise interaction

$$
\begin{aligned}
\mathcal{A}_3 \;=\; & 2U_4^{dd'}\Big\{i\big[f^{ad'e'}(U_1t^a)_{ij}(t^dt^eU_2)_{kl} - f^{ade}(t^aU_1)_{ij}(U_2t^{e'}t^{d'})_{kl}\big] \\
& \times\Big[\mathcal{J}_{1245}\ln\frac{z_{14}^2}{z_{15}^2} + (J_{2145}-J_{2154})\ln\frac{z_{24}^2}{z_{25}^2}\Big](U_5-U_2)^{ee'} \\
& + i\big[f^{ad'e'}(t^dt^eU_1)_{ij}(U_2t^a)_{kl} - f^{ade}(U_1t^{e'}t^{d'})_{ij}(t^aU_2)_{kl}\big] \\
& \times\Big[\mathcal{J}_{2145}\ln\frac{z_{24}^2}{z_{25}^2} + (J_{1245}-J_{1254})\ln\frac{z_{14}^2}{z_{15}^2}\Big](U_5-U_1)^{ee'}\Big\}
\end{aligned}
$$

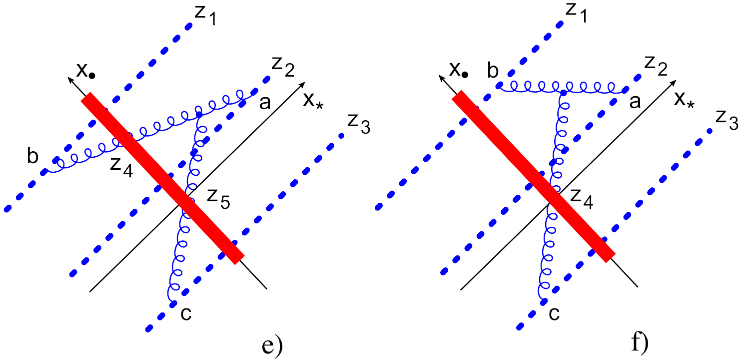
$$
\begin{aligned}
\mathcal{J}_{1245} \;\equiv\; & \mathcal{J}(z_1,z_2,z_4,z_5) \\
= \; & \frac{(z_{24},z_{25})}{z_{24}^2z_{25}^2z_{45}^2} - \frac{2(z_{24},z_{45})(z_{15},z_{25})}{z_{24}^2z_{25}^2z_{15}^2z_{45}^2} + \frac{2(z_{25},z_{45})(z_{14},z_{24})}{z_{14}^2z_{24}^2z_{25}^2z_{45}^2} - 2\frac{(z_{14},z_{24})(z_{15},z_{25})}{z_{14}^2z_{15}^2z_{24}^2z_{25}^2}
\end{aligned}
$$

Pairwise interaction

$$\mathcal{B}_1 = 2\ln\frac{z_{14}^2}{z_{12}^2}\ln\frac{z_{24}^2}{z_{12}^2}$$
$$\times\Big\{(U_4-U_1)^{ab}i\big[f^{bde}(t^aU_1t^d)_{ij}(U_2t^e)_{kl}+f^{ade}(t^eU_1t^b)_{ij}(t^dU_2)_{kl}\big]\Big[\frac{(z_{14},z_{24})}{z_{14}^2z_{24}^2}-\frac{1}{z_{14}^2}\Big]$$
$$+(U_4-U_2)^{ab}i\big[f^{bde}(U_1t^e)_{ij}(t^aU_2t^d)_{kl}+f^{ade}(t^dU_1)_{ij}(t^eU_2t^b)_{kl}\big]\Big[\frac{(z_{14},z_{24})}{z_{14}^2z_{24}^2}-\frac{1}{z_{24}^2}\Big]\Big\}$$

$$\mathcal{B}_2 = \big[2U_4^{ab}-U_1^{ab}-U_2^{ab}\big]\big[(t^aU_1)_{ij}(U_2t^b)_{kl}+(U_1t^b)_{ij}(t^aU_2)_{kl}\big]$$
$$\times\Big\{\frac{(z_{14},z_{24})}{z_{14}^2z_{24}^2}\Big[\frac{11}{3}\ln z_{12}^2\mu^2+\frac{67}{9}-\frac{\pi^2}{3}\Big]+\frac{11}{3}\Big[\frac{1}{2z_{14}^2}\ln\frac{z_{24}^2}{z_{12}^2}+\frac{1}{2z_{24}^2}\ln\frac{z_{14}^2}{z_{12}^2}\Big]\Big\}$$

Triple interaction



$$\mathcal{J}_{12345} \equiv \mathcal{J}(z_1, z_2, z_3, z_4, z_5) = -\frac{2(z_{14}, z_{34})(z_{25}, z_{35})}{z_{14}^2 z_{25}^2 z_{34}^2 z_{35}^2}$$

$$-\frac{2(z_{14}, z_{45})(z_{25}, z_{35})}{z_{14}^2 z_{25}^2 z_{35}^2 z_{45}^2} + \frac{2(z_{25}, z_{45})(z_{14}, z_{34})}{z_{14}^2 z_{25}^2 z_{34}^2 z_{45}^2} + \frac{(z_{14}, z_{25})}{z_{14}^2 z_{25}^2 z_{45}^2}$$

Triple interaction

$$
\begin{aligned}
&\frac{d}{d\eta}(U_1)_{ij}(U_2)_{kl}(U_3)_{mn}\\
&= \; i\frac{\alpha_s^2}{2\pi^4}\int d^2z_4 d^2z_5 \, \Big\{\mathcal{J}_{12345}\ln\frac{z_{34}^2}{z_{35}^2}\\
&\times f^{cde}\Big[(t^aU_1)_{ij}(t^bU_2)_{kl}(U_3t^c)_{mn}(U_4-U_1)^{ad}(U_5-U_2)^{be}\\
&-(U_1t^a)_{ij}(U_2t^b)_{kl}(t^cU_3)_{mn}(U_4-U_1)^{da}(U_5-U_2)^{eb}\Big]\\
&+\mathcal{J}_{32145}\ln\frac{z_{14}^2}{z_{15}^2}\\
&\times f^{ade}\Big[(U_1t^a)_{ij}(t^bU_2)_{kl}(t^cU_3)_{mn}(U_4-U_3)^{cd}(U_5-U_2)^{be}\\
&-(t^aU_1)_{ij}\otimes(U_2t^b)_{kl}(U_3t^c)_{mn}(U_4^{dc}-U_3^{dc})(U_5^{eb}-U_2^{eb})\Big]\\
&+\mathcal{J}_{13245}\ln\frac{z_{24}^2}{z_{25}^2}\\
&\times f^{bde}\Big[(t^aU_1)_{ij}(U_2t^b)_{kl}(t^cU_3)_{mn}(U_4-U_1)^{ad}(U_5-U_3)^{ce}\\
&-(U_1t^a)_{ij}(t^bU_2)_{kl}(U_3t^c)_{mn}(U_4-U_1)^{da}(U_5-U_3)^{ec}\Big]
\end{aligned}
\qquad (1)
$$

Conclusions

- High-energy operator expansion in color dipoles works at the NLO level.

Conclusions

- High-energy operator expansion in color dipoles works at the NLO level.
- The NLO BK kernel in for the evolution of conformal composite dipoles in $\mathcal{N} = 4$ SYM is Möbius invariant in the transverse plane.
- The NLO BK kernel agrees with NLO BFKL equation.
- The correlation function of four Z^2 operators is calculated at the NLO order.
- It gives the anomalous dimensions of gluon light-ray operators at "the BFKL point" $j \to 1$
- NLO photon impact factor is calculated.
- NLO hierarchy of Wilson-line evolution is obtained